University of California, Berkeley
Physics 110B, Fall 2003 (*Strovink*)

PROBLEM SET 8

1.
Consider a medium with uniform fixed dielectric constant ϵ, permeability μ, and volume conductivity σ.
(a.)
Taking the curl of the two Maxwell equations which themselves involve the curl, and using Ohm's law ($\vec{J} = \sigma\vec{E}$) and the two other Maxwell equations where appropriate, derive the wave equations

$$\Big(\nabla^2 - \frac{\partial^2}{v^2\,\partial t^2}\Big)\vec{B} = \sigma\mu\frac{\partial\vec{B}}{\partial t}$$
$$\Big(\nabla^2 - \frac{\partial^2}{v^2\,\partial t^2}\Big)\vec{E} = \sigma\mu\frac{\partial\vec{E}}{\partial t} + \frac{1}{\epsilon}\nabla\rho_f\ ,$$

where the phase velocity^2 $v^2 \equiv \frac{1}{\epsilon\mu}$ and ρ_f is the volume free charge density.
(b.)
In the wave equation for $\vec{B}$ derived in (a.), substitute

$$\vec{B}(\vec{r},t) = \mathrm{Re}\Big(\vec{\tilde{B}}\exp\big(i(\vec{\tilde{k}}\cdot\vec{r} - \omega t)\big)\Big)\ ,$$

where $\vec{\tilde{B}}$ and $\vec{\tilde{k}}$ are complex (vector) constants. Show that

$$\frac{\tilde{k}^2}{(\omega/v)^2} = 1 + i\beta\ ,$$

where $\beta \equiv \frac{\sigma}{\epsilon\omega}$.

2. Please refer to the notation of the previous problem.
(a.)
Write $\tilde{k} \equiv k + i\kappa$, where k and κ are real. Show that

$$\frac{k}{\omega/v} = \sqrt{\frac{\sqrt{1+\beta^2}+1}{2}}$$
$$\frac{\kappa}{\omega/v} = \sqrt{\frac{\sqrt{1+\beta^2}-1}{2}}$$

are solutions to the equation in part (b.) of the previous problem.
(b.)
κ^{-1}, the inverse of the imaginary part of $\tilde{k}$, is called the *skin depth*. Show that the skin depth approaches

$$\sqrt{\frac{2}{\mu\sigma\omega}}\ \text{ when } \beta \gg 1$$
$$\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}\ \text{ when } \beta \ll 1\ .$$

3.
Please refer to the notation and results of the previous two problems. At normal incidence at the interface between two dissimilar materials 1 and 2, the (complex) electric field amplitude reflected back into material 1 is expressed as a (complex ratio) $\tilde{\mathcal{R}}$ to the (complex) incident amplitude. By matching boundary conditions for the electric and magnetic fields, $\tilde{\mathcal{R}}$ is routinely found to be given by the standard result

$$\tilde{\mathcal{R}} = \frac{\tilde{Z}_1^{-1} - \tilde{Z}_2^{-1}}{\tilde{Z}_1^{-1} + \tilde{Z}_2^{-1}}\ ,$$

where

$$\tilde{Z}^{-1} \equiv \frac{\tilde{k}}{\mu\omega}$$

is the medium's (complex) *admittance*. Consider the case in which material 1 is an insulator and material 2 is a conductor.
(a.)
If material 2 is an *excellent* conductor ($\beta \gg 1$), show that $\tilde{\mathcal{R}} \to -1$ regardless of the (finite) values taken by $\epsilon_{1,2}$ and $\mu_{1,2}$.
(b.)
Suppose that, if both materials were insulators, they would have equal admittance ($\sqrt{\epsilon_1/\mu_1} = \sqrt{\epsilon_2/\mu_2}$). Suppose further that material 2 is a *poor* conductor ($\beta \ll 1$). Show that $\tilde{\mathcal{R}} \to -i\beta/4$.

4.
Please refer to the notation and results of the previous three problems. In a relatively more microscopic and detailed treatment, we assume that N valence electrons/m^3 of charge $-e$ and mass m move in a potential well with effective spring constant $m\omega_0^2$ and damping coefficient γm. We define the complex dielectric constant

$$\tilde{\epsilon} \equiv \epsilon(1+i\beta) = \frac{\tilde{k}^2}{\mu\omega^2} \ .$$

For not-too-dense media in which the electric field felt by the electron is approximately the same as the average field, in class we derived

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \ ,$$

where the plasma frequency^2 is

$$\omega_p^2 \equiv \frac{Ne^2}{m\epsilon_0} \ .$$

(a.)
Represent a *good conductor* by $\omega_0 = 0$ (unbound) and $\gamma \gg \omega$ (overdamped). Show that the conductivity

$$\sigma \approx \frac{\epsilon_0 \omega_p^2}{\gamma} \ .$$

(b.)
Represent the *ionosphere* by $\omega_0 = 0$ (unbound), and $\gamma \ll \omega$ (underdamped). Specialize to AM radio waves, for which $\omega < \omega_p$. Show that $|\tilde{\mathcal{R}}| \approx 1$, *i.e.* that AM radio waves are nearly fully reflected by the ionosphere. (At dusk, the ionosphere drops to sufficiently low altitude that reflection off it enables AM stations hundreds of miles away to be received.)

5.
Please refer to the results and notation of the previous problem. Consider an *insulating solid* (ultraviolet ω_0, $\omega_p > \omega_0$) that is *transparent in the visible* ($\gamma \ll \omega_0$). Assume that the solid has negligible magnetic properties ($\mu \approx \mu_0$). For convenience, work with the *complex refractive index*

$$\tilde{n} \equiv \frac{\tilde{k}}{\omega/c} \ .$$

(a.)
For ω well above resonance ($\omega - \omega_0 \gg \gamma$), show that

$$\tilde{n}^2 \approx 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2} \ .$$

(b.)
Writing $\tilde{n} = n + i\eta$, where n and η are real, show that either n or η, but not both, must be $\ll 1$.
(c.)
For ω well above resonance, but still below $\sqrt{\omega_0^2 + \omega_p^2}$, show that n must be $\ll 1$.
(d.)
For the conditions of part (c.), show that $|\tilde{\mathcal{R}}| \approx 1$, *i.e.* that visibly transparent insulators are nearly fully reflecting over a band of ultraviolet frequencies.

6.
Griffiths Problem 9.11.

7. Jones vectors.
For a plane transverse wave propagating in the $\hat{z}$ direction through a (not necessarily insulating) material with constant ϵ and μ, a (co)sinusoidal solution is represented by

$$\vec{E}(\vec{r},t) = \mathrm{Re}\Big(\vec{E}_0(x,y)\, e^{i(\tilde{k}z - \omega t)}\Big)$$
$$\vec{H}(\vec{r},t) = \mathrm{Re}\Big(\vec{H}_0(x,y)\, e^{i(\tilde{k}z - \omega t)}\Big) \ ,$$

where $\tilde{k}$ is the (not necessarily real) "wave vector" – here a scalar because we know it is directed along $\hat{z}$. Faraday's law causes $\vec{H}_0$ to be completely determined by $\vec{E}_0$:

$$\vec{H}_0 \equiv \tilde{Z}^{-1} \hat{z} \times \vec{E}_0 = \frac{\tilde{k}}{\mu\omega} \hat{z} \times \vec{E}_0 \ ,$$

so we focus on $\vec{E}_0$ as the sole independent variable. For a transverse wave $\vec{E}_0$ has no z component. Here we assume that the phase relationship between E_{0x} and E_{0y} is *fixed* – the wave is *fully* *polarized*. Then $\vec{E}_0$ is a complex transverse vector, completely specified by four components. In the Jones convention, all information carried by $\vec{E}_0$ except for its magnitude is written as a 2×1 column vector with the x component on top:

$$\begin{aligned} \vec{E}_0 &= \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} \\ &\equiv \frac{1}{\sqrt{|\alpha^2| + |\beta^2|}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |\vec{E}_0| \\ &\equiv \vec{J} |\vec{E}_0| \ , \end{aligned}$$

where $\vec{J}$ is the *Jones vector*. Jones vectors are defined only within an overall phase (because the absolute phase of an optical-frequency EM wave can't conveniently be measured); therefore one has the freedom to set α equal to unity (unless it vanishes, in which case β is set to unity). The above form involving the complex constants α and β is a general Jones vector, corresponding to elliptical polarization. More common Jones vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \ ,$$

corresponding, respectively, to linear x, linear y, RH circular, and LH circular polarization.

(a.)

At $z = 0$, show (counterintuitively!) that the electric field vector for RH polarized light precesses *clockwise* around $\hat{z}$, *i.e.* it precesses according to the LH rule.

(b.)

Suppose that a particular state of elliptical polarization has nonvanishing x and y electric field components. Then, within an arbitrary overall phase, it may be represented by the Jones vector

$$\vec{J}_1 = \frac{1}{\sqrt{1 + |\beta|^2}} \begin{pmatrix} 1 \\ \beta \end{pmatrix} \ ,$$

where β is a complex constant. You wish to characterize this state of polarization as "RH elliptical" or "LH elliptical", depending on whether (at $z = 0$) the electric field vector precesses clockwise or counterclockwise around $\hat{z}$. What property of β would you use to decide whether this state is RH or LH elliptical?

(c.)

For the conditions of part (b.), decompose $\vec{J}_1$ into a linear sum (with real coefficients) of a wave with linear polarization plus a wave with RH circular polarization. Perform this same task with "RH" replaced by "LH". If you are successful in both tasks, you might wonder whether there really exists a unique association of RH or LH behavior with $\vec{J}_1$. Would this concern invalidate your answer to (b.)?

8. Irradiance and Jones vectors.

Consider two transverse plane waves A and B which move in vacuum and are combined together (*i.e.* by a Michelson interferometer). The beams have complex electric fields

$$\begin{aligned} \begin{pmatrix} E_{0x}^A \\ E_{0y}^A \end{pmatrix} &= \frac{|\vec{E}_0^A|}{\sqrt{|\alpha|^2 + |\beta|^2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{pmatrix} E_{0x}^B \\ E_{0y}^B \end{pmatrix} &= \frac{|\vec{E}_0^B|}{\sqrt{|\gamma|^2 + |\delta|^2}} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \ . \end{aligned}$$

Express the combined irradiance

$$I_{A+B} \equiv \langle \vec{S}_{A+B} \cdot \hat{z} \rangle \ ,$$

where $\vec{S}$ is the Poynting vector and $\langle\ \rangle$ is a time average, as a function of the complex constants α, β, γ, δ, and the uncombined irradiances I_A and I_B of the individual beams.